MATHEMATICS

ON REARRANGEMENTS OF INFINITE SERIES

BY

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1. *Introduction*

In a previous paper [1]) (which will be designated hereafter by [I])the author has shown that for every sequence $a_1, a_2, a_3, \ldots$ of positive terms with $\sum_{n=1}^{\infty} a_n = \infty$, and $a_n \to 0$, it is possible to find a sequence $b_1, b_2, b_3, \ldots$ which is a reordering of the terms of a subsequence of $a_1, a_2, a_3, \ldots$ and which has the following property: if $c_1, c_2, c_3, \ldots$ is a subsequence of $b_1, b_2, b_3, \ldots$ with $\sum_{n=0}^{\infty} c_n = \infty$ then also $\sum_{n=0}^{\infty} |c_n - c_{n+1}| = \infty$. In fact it was shown that for any subsequence $c_0, c_1, c_2, \ldots$ of $b_0, b_1, b_2, \ldots$,

$$\sum_{j=0}^{n=1} |c_j - c_{j+1}| > 2 \sum_{k=0}^{n} c_k - 5b^*$$

where b^* denotes the largest of the terms $b_1, b_2, b_3, \ldots$. In this formula 2 is the best constant, 5 is not. It is the purpose of the present paper to answer two questions intimately related to the above.

The first question, which arises quite naturally, is: under what circumstances may the sequence $b_1, b_2, b_3, \ldots$ be chosen so as to exhaust the terms of $a_1, a_2, a_3, \ldots$? Professor J. G. VAN DER CORPUT has conjectured that this would be possible if and only if the number of terms of $a_1, a_2, a_3, \ldots$ greater than $1/x$ were $o(x)$ as $x \to \infty$. This result will be established in sec. 2.

The second question is: can similar results to the above be established for higher differences than the first? That is, with all the hypotheses and notations of the first paragraph can we show that

$$\sum_{n=1}^{\infty} |c_n - 2c_{n+1} + c_{n+2}| = \infty, \quad \sum_{n=1}^{\infty} |c_n - 3c_{n+1} + 3c_{n+2} - c_{n+3}| = \infty, \text{ etc.?}$$

This question is answered affirmatively in section 3 where it is in fact shown that with the hypotheses on $a_1, a_2, a_3, \ldots$ as stated there exists a reordered subsequence $b_1, b_2, b_3, \ldots$ of $a_1, a_2, a_3, \ldots$ such that for any

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[1]) On Sequences with Divergent Total Variation, these Proceedings ser. A, **58**, No. 2, pp. 178–190.

subsequence $c_1, c_2, c_3, \ldots$ of $b_1, b_2, b_3, \ldots$ and for numbers $\alpha_0, \alpha_1, \ldots, \alpha_q$, real or complex,

$$\sum_{j=1}^{n-q} |\alpha_0 c_n + \alpha_1 c_{n+1} + \ldots + \alpha_q c_{n+q}| >$$
$$(|\alpha_0| + |\alpha_1| + \ldots + |\alpha_q|) \sum_{j=1}^{n} c_j - q(q+1)\, \alpha^* \, b^*$$

where b^* is the largest of the numbers $b_1, b_2, \ldots$ and α^* is the largest of the numbers $|\alpha_0|, |\alpha_1|, \ldots, |\alpha_q|$.

2. *Complete Rearrangements*

In this section $a_1, a_2, a_3, \ldots$ will denote a sequence of positive numbers with $\sum_{n=1}^{\infty} a_n = \infty$ and $a_n \to 0$ as $n \to \infty$. Furthermore we will define for $x > 0$

$$N(x) = \text{the number of integers } n \text{ for which } a_n \geqslant 1/x.$$

Noting that $N(x)$ tends monotonely to ∞ as $x \to \infty$, we will now establish the following

LEMMA 2.1. *Suppose $N(x) \neq o(x)$. Let $m > 0$, $p > 1$ and $0 < K < \limsup_{x \to \infty} N(x)/x$. Then there is a number $t > m$ such that*

$$N(pt) - N(t) > K/2\,(p-1)\,t.$$

Proof. Suppose false and let $t > pm + 2/K\ N(pm)$ with

$$N(t)/t > K \tag{2.1}$$

so that

$$(K/2)/t > N(pm). \tag{2.2}$$

Now let n be the integer (greater than 0) for which

$$m \leqslant t/p^n < pm. \tag{2.3}$$

Using (2.1), (2.3) and our supposition we obtain

$$Kt < N(t) = \sum_{j=0}^{n-1} \left[N\left(\frac{t}{p^j}\right) - N\left(\frac{t}{p^{j+1}}\right) \right] + N\left(\frac{t}{p^n}\right)$$
$$\leqslant \sum_{j=0}^{n-1} \frac{K}{2} (p-1) \frac{t}{p^{j+1}} + N(pm)$$
$$= \frac{Kt}{2} \sum_{j=0}^{n-1} \left(\frac{1}{p^j} - \frac{1}{p^{j+1}} \right) + N(pm)$$
$$< \frac{K}{2} t + N(pm).$$

Hence $(K/2)/t < N(pm)$ which contradicts (2.2) and thereby completes the proof.

DEFINITION 2.1. We say that $b_1, b_2, b_3, \ldots$ is a complete rearrangement of $a_1, a_2, a_3, \ldots$ if and only if there is a $1-1$ function h on the positive integers onto the positive integers such that $b_n = a_{h(n)}$.

THEOREM 2.1. *If $N(x) \neq o(x)$ and $b_1, b_2, b_3, \ldots$ is a complete rearrangement of $a_1, a_2, a_3, \ldots$ then there is a subsequence $c_1, c_2, c_3, \ldots$ of $b_1, b_2, b_3, \ldots$ with $\sum_{k=1}^{\infty} c_k = \infty$ while $\sum_{k=1}^{\infty} |c_k - c_{k+1}| < \infty$.*

Proof. Using the results and the notation of the above lemma, choose a number $x_1 > 0$, for which

$$N(2x_1) - N(x_1) < (K/2)\, x_1.$$

Accordingly, letting n_1 be the integer with

$$(K/2)x_1 \leqslant n_1 < (K/2)x_1 + 1,$$

we may choose as the first n_1 terms of the sequence $c_1, c_2, c_3, \ldots,$ terms of $b_1, b_2, b_3, \ldots$ which are $\leqslant 1/x_1$ and $> 1/(2x_1)$. Suppose that terms $c_1, c_2, \ldots, c_{n_j}$ have been chosen. Let r_j be such a positive integer that:

$$\left\{\begin{array}{l} b_{r_j} < \min\, \{c_1, c_2, \ldots, c_{n_j}\}; \\ \text{if } b_l \leqslant b_{r_j}, \text{ then } l \geqslant r_j. \end{array}\right. \tag{2.4}$$

Now using the above lemma choose a number $x_{j+1} \geqslant \max\, \{1/b_{r_j}, j(j+1)\}$ so that

$$N\left(\frac{j+1}{j}\, x_{j+1}\right) - N(x_{j+1}) > \frac{K}{2}\left(\frac{j+1}{j} - 1\right) x_{j+1} = \frac{K}{2} \cdot \frac{x_{j+1}}{j}.$$

Accordingly, letting n_{j+1} be the integer with

$$\frac{K}{2} \cdot \frac{x_{j+1}}{j} \leqslant n_{j+1} - n_j < \frac{K}{2} \cdot \frac{x_{j+1}}{j} + 1,$$

we choose for the terms $c_{n_j+1}, c_{n_j+2}, \ldots, c_{n_{j+1}}$, terms of $b_1, b_2, b_3, \ldots$ which are $\leqslant \dfrac{1}{x_{j+1}}$ and $> \dfrac{1}{\frac{j+1}{j}\, x_{j+1}}$.

Now

$$\sum_{l=n_j+1}^{n_{j+1}} c_l \geqslant \frac{K}{2} \cdot \frac{x_{j+1}}{j} \cdot \frac{1}{\frac{j+1}{j}\, x_{j+1}} = \frac{K}{2} \cdot \frac{1}{j+1}. \tag{2.5}$$

Moreover

$$
(2.6)\quad \left\{
\begin{aligned}
\sum_{l=n_j+1}^{n_{j+1}-1} |c_l - c_{l+1}| &\leqslant \sum_{l=n_j+1}^{n_{j+1}} \left(\frac{1}{x_{j+1}} - \frac{1}{\frac{j+1}{j} x_{j+1}} \right) \\
&= (n_{j+1} - n_j) \left(\frac{1}{x_{j+1}} - \frac{1}{\frac{j+1}{j} x_{j+1}} \right) \\
&= (n_{j+1} - n_j) \frac{1}{x_{j+1}} \cdot \frac{1}{j+1} \\
&< \left(\frac{K}{2} \cdot \frac{x_{j+1}}{j} + 1 \right) \frac{1}{x_{j+1}} \cdot \frac{1}{j+1} \\
&= \frac{K}{2} \cdot \frac{1}{j(j+1)} + \frac{1}{x_{j+1}} \cdot \frac{1}{j+1} \\
&\leqslant \frac{K}{2} \cdot \frac{1}{j(j+1)} + \frac{1}{j(j+1)} \cdot \frac{1}{j+1} \\
&\leqslant \left(\frac{K}{2} + 1 \right) \frac{1}{j(j+1)} .
\end{aligned}
\right.
$$

Finally, letting $n_0 = 0$, we have by (2.5)

$$
\sum_{l=1}^{\infty} c_l = \sum_{j=0}^{\infty} \sum_{l=n_j+1}^{n_j+1} c_l \geqslant \sum_{j=0}^{\infty} \frac{K}{2} \cdot \frac{1}{j+1} = \infty .
$$

On the other hand (2.4) together with (2.6) yields

$$
\begin{aligned}
\sum_{l=1}^{\infty} |c_l - c_{l+1}| &= \sum_{j=0}^{\infty} \sum_{l=n_j+1}^{n_j+1} |c_l - c_{l+1}| + \sum_{j=1}^{\infty} |c_{n_j} - c_{n_j+1}| \\
&\leqslant \left(\frac{K}{2} x_1 + 1 \right) \cdot \frac{1}{x_1} + \sum_{j=1}^{\infty} \left(\frac{K}{2} + 1 \right) \frac{1}{j(j+1)} + b^* < \infty
\end{aligned}
$$

where b^* denote the largest term of $b_1, b_2, b_3, \ldots$. This completes the proof.

Now we consider the case that $N(x) = o(x)$. Accordingly we assume in the remainder of this section that $a_1, a_2, \ldots$ is a sequence satisfying together with the conditions given in the first paragraph of the section, the condition

$$
N(x) = o(x).
$$

We make the

Definition 2.2. For $x > 0$, we let

$L(x) =$ the integer n for which $1/2^{n+1} < x \leqslant 1/2^n$. If $L(x) = r$ we say that x has level r.

We proceed to construct a sequence related to $a_1, a_2, \ldots$ which will be used throughout the remainder of this section.

Construction. Let $\varepsilon > 0$ and choose $\varepsilon_0, \varepsilon_1, \ldots$ so that

$$(2.7)\qquad \begin{cases} \varepsilon_0 = \max\limits_{x>0} \dfrac{N(x)}{x} \\ \varepsilon_n > 0 \text{ for } n = 1, 2, \ldots, \text{ and } \sum\limits_{n=1}^{\infty} \varepsilon_n = \varepsilon. \end{cases}$$

Next choose integers $0^*, 1^*, 2^*, \ldots$ so that

$$(2.8)\qquad \begin{cases} 0^* < 1^* < 2^* < \ldots, \\ 0^* = \min\limits_{k=1}^{\infty} L(a_k), \\ \text{if } x \geqslant 2^{n^*} \text{ then } N(x) < \varepsilon_n x \text{ for } n = 1, 2, 3, \ldots . \end{cases}$$

(Observe that from the definition of ε_0 in (2.7) the statement in (2.8) also holds for $n = 0$.)

Now for each integer $n \geqslant 0$ we construct a finite sequence

$$(2.9)\qquad d_1{}^n, d_2{}^n, \ldots, d_{N_n}{}^n$$

according to the following rules:

(1) Each term $d_k{}^n$ must satisfy

$$n^* \leqslant L(d_k{}^n) < (n+1)^*.$$

(2) Choose $[\varepsilon_n 2^{n^*+1} + 1]$ terms with level n^{n^*} selecting these terms from $a_1, a_2, \ldots$ as long as such terms are available, thereafter choosing them arbitrarily. Order these terms in any way. These are the terms of (2.9) with level n^*.

(3) Next choose $2[\varepsilon_n \cdot 2^{n^*+1} + 1]$ terms with level $n^* + 1$ selecting these terms from $a_1, a_2, \ldots$ as long as such terms are available, thereafter choosing them arbitrarily. These are the terms of (2.9) with level $n^* + 1$. Now order these terms in any way and place one before the first term previously chosen, two between each pair of terms previously chosen and one term after the last term previously chosen.

(4) When the terms of (2.9) with levels less than r have been chosen, where $n^* + 1 < r < (n+1)^*$, we choose 2^{r-n^*} $[\varepsilon_n \cdot 2^{n^*+1} + 1]$ terms with level r, choosing these terms from $a_1, a_2, a_3, \ldots$ as long as such terms are available, thereafter choosing them arbitrarily. These are the terms of (2.9) with level r. Now order these terms in any way and place one term before the first term previously chosen, one term between each pair of consecutive terms previously chosen and one term after the last term previously chosen.

(5) The above process is to be terminated when the level of terms chosen reaches $(n+1)^* - 1$.

(6) The terms chosen in the above are to be designated in the indicated order as

$$d_1{}^n, d_2{}^n, \ldots, d_{N_n}{}^n.$$

(7) The terms

$$d_1{}^0, d_2{}^0, \ldots, d_{N_0}{}^0, d_1{}^1, d_2{}^1, \ldots, d_{N1}{}^1, \ldots, d_1{}^n, d_2{}^n, \ldots, d_{N_n}{}^n, \ldots$$

are to be designated in this order as

$$d_1, d_2, d_3, \ldots .$$

The properties of the sequences constructed above which we will need are enumerated in the following remarks.

REMARK 2.1. The sequence $d_1, d_2, d_3, \ldots$ exhausts the terms of $a_1, a_2, a_3, \ldots$.

Proof. The number of terms of $a_1, a_2, a_3, \ldots$ with level equal to r where $n^* \leqslant r < (n+1)^*$ is less than or equal to the number of terms of $a_1, a_2, a_3, \ldots$ which are greater than or equal to $1/2^{r+1}$. That is, the number of terms of level r is less than or equal to $N(2^{r+1})$. By (2.8) we have

$$N(2^{r+1}) < \varepsilon_n \cdot 2^{r+1}.$$

In the above construction, the number of terms chosen with level r was $2^{r-n^*}\ [\varepsilon_n \cdot 2^{n^*+1}+1]$ and

$$2^{r-n^*}\ [\varepsilon_n \cdot 2^{n^*+1}+1] \geqslant 2^{r-n^*}\ \varepsilon_n \cdot 2^{n^*+1} = \varepsilon_n \cdot 2^{r+1}.$$

REMARK 2.2. If $\max\,(L(d_i{}^n), L(d_j{}^n)) = r$ where $n^* \leqslant r < (n+1)^*-1$ then there are between $d_i{}^n$ and $d_j{}^n$ at least $2^{(n+1)^*-(r+1)}$ terms of $d_1{}^n, d_2{}^n, \ldots, d_{N_n}{}^n$ with level $(n+1)^*-1$.

Proof. Between $d_i{}^n$ and $d_j{}^n$ there are at least one term with level $r+1$, at least 2 terms with level $r+2, \ldots,$ at least $2^{(n+1)^*-(r+1)}$ terms with level $(n+1)^*-1$.

REMARK 2.3. If $n^* \leqslant r \leqslant (n+1)^*-1$ then the sum S_r of terms of $d_1{}^n, d_2{}^n, \ldots, d_{N_n}{}^n$ with level r satisfies

$$\varepsilon_n \leqslant S_r \leqslant 2\varepsilon_n + 2^{-n^*}.$$

Proof. There are $2^{r-n^*}\ [\varepsilon_n 2^{n^*+1}+1]$ such terms and each term is less than or equal to 2^{-r} and greater than $2^{-(r+1)}$. Therefore

$$\varepsilon_n \leqslant 2^{r-n^*}\ [\varepsilon_n\ 2^{n^*+1}+1]\ 2^{-(r+1)} \leqslant S_r \leqslant 2^{r-n^*}\ [\varepsilon_n\ 2^{n^*+1}+1]\ 2^{-r} \leqslant 2\varepsilon_n + 2^{-n^*}.$$

DEFINITION 2.3. We define $d_0{}^n = d_{N_n+1}{}^n = 0$.

DEFINITION 2.4. For k and l integers with $0 \leqslant k \leqslant l \leqslant N_n+1$ we define

$$\sum_{i=k}^{l}{}^* \ d_i{}^n = \sum_{i=k}^{l} \beta_i{}^n\, d_i{}^n - {}^1\!/_2(\beta_k{}^n\, d_k{}^n + \beta_l{}^n\, d_l{}^n)$$

where

$$\beta_i{}^n = \begin{cases} 1 \text{ if } L(d_i{}^n) = (n+1)^*-1 \\ 0 \text{ if } L(d_i{}^n) \neq (n+1)^*-1. \end{cases}$$

It is evident from these definitions that for $0 \leqslant k \leqslant l \leqslant m \leqslant N_n + 1$,

$$\sum_{i=k}^{l}{}^{*}\, d_i^n + \sum_{i=l}^{m}{}^{*}\, d_i^n = \sum_{i=k}^{m}{}^{*}\, d_i^n \tag{2.10}$$

and that

$$\sum_{i=0}^{N_n+1}{}^{*}\, d_i^n = \sum_{i=1}^{N_n} d_i^n. \tag{2.11}$$

$L(d_i^n) = (n+1)^* - 1$

LEMMA 2.2. If $L(d_i^n) \geqslant L(d_j^n)$ with $i \neq j$ then

$$\sum_{k=\min(i,j)}^{\max(i,j)}{}^{*} \quad d_k^n \geqslant \tfrac{1}{4}\, d_i^n. \tag{2.12}$$

Proof. If $L(d_i^n) = (n+1)^* - 1$ the conclusion is obvious since the term $^1/_2\, d_i^n$ is included in the sum on the left. If $L(d_i^n) = r < (n+1)^* - 1$ then the conclusion follows from (2.15) and (2.14) below.

From the definition of L we see that

$$d_i^n \leqslant 2^{-r} \tag{2.13}$$

and that each term d_k^n in the sum on the left of (2.12) satisfies

$$d_k^n > 2^{-(n+1)^*}. \tag{2.14}$$

By remark 2.2 the number of terms in the sum on the left hand side of (2.12) is greater than or equal to

$$2^{(n+1)^* - (r+1)}.$$

Using this fact together with (2.13) and (2.14) we have

$$\sum_{k=\min(i,j)}^{\max(i,j)}{}^{*}\, d_k^n \geqslant 2^{(n+1)^* - (r+2)} \cdot 2^{-(n+1)^*} = \tfrac{1}{4}\, 2^{-r} \geqslant \tfrac{1}{4}\, d_i^n.$$

LEMMA 2.3.

$$\sum_{k=0}^{i}{}^{*}\, d_k^n \geqslant \tfrac{1}{4} d_i^n; \quad \sum_{k=i}^{N_n+1}{}^{*}\, d_k^n \geqslant \tfrac{1}{4}\, d_i^n.$$

The proof is similar to that of Lemma 2.1.

LEMMA 2.4. *If $c_1, c_2, \ldots, c_s$ is a finite subsequence of $d_1^n, d_2^n, \ldots, d_{N_n}^n$ and if we let $h(k)$ be the strictly monotone integer valued function of k such that*

$$c_k = d_{h(k)}^n$$

then

$$\sum_{j=h(1)}^{h(s)}{}^{*} \quad d_j^n \geqslant \begin{cases} \tfrac{1}{4} \sum\limits_{k=1}^{s-1} c_k \text{ if } L(c_1) \geqslant L(c_2) \geqslant \ldots \geqslant L(c_s), \\ \tfrac{1}{4} \sum\limits_{k=2}^{s} c_k \text{ if } L(c_1) \leqslant L(c_2) \leqslant \ldots \leqslant L(c_s). \end{cases}$$

Proof. If $L(c_1) \geqslant L(c_2) \geqslant \ldots \geqslant L(c_s)$ then

$$\sum_{j=h(1)}^{h(s)} {}^{*} d_j^n = \sum_{k=1}^{s-1} \sum_{j=h(k)}^{h(k+1)} {}^{*} d_j^n \geqslant \sum_{k=1}^{s-1} c_k,$$

the equality following from (2.10) and the inequality from Lemma 2.2. The other case is established in the same way.

We now introduce same terminology which will be used in Lemma 2.5 and Theorem 2.2. We let $c_1, c_2, \ldots, c_s$ be an arbitrary subsequence of $d_1^n, d_2^n, \ldots, d_{N_n}^n$ and let $e_1, e_2, \ldots, e_q$ be a subsequence of $c_1, c_2, \ldots, c_s$ satisfying the following conditions:

If $e_k = c_{\alpha(k)}$, $k = 1, 2, \ldots, q$,

then

$$\begin{array}{lllll}
\text{I} & L(c_j) \leqslant L(c_{j+1}) & \text{for} & 1 \leqslant j < \alpha(1), & \\
 & L(c_j) \geqslant L(c_{j+1}) & \text{for} & \alpha(q) \leqslant j < s; & \\
\text{II} & L(c_j) \geqslant L(c_{j+1}) & \text{for} & \alpha(k) \leqslant j < \alpha(k+1) & \text{when } k \text{ is odd,} \\
 & L(c_j) \leqslant L(c_{j+1}) & \text{for} & \alpha(k) \leqslant j < \alpha(k+1) & \text{when } k \text{ is even;} \\
\text{III} & L(e_k) \neq L(e_{k+1}) & \text{for} & 1 \leqslant k < q, & \\
 & L(c_1) \neq L(e_1) & \text{unless} & \alpha(1) = 1, & \\
 & L(c_s) \neq L(e_q) & \text{unless} & \alpha(q) = s. &
\end{array}$$

It is clear that q is odd.

Lemma 2.5. *If $c_1, c_2, \ldots, c_s$ is a subsequence of $d_1^n, d_2^n, \ldots, d_{N_n}^n$ and $e_1, e_2, \ldots, e_q$ is constructed as above then*

$$\sum_{\substack{i=1 \\ L(d_i^n)=(n+1)^*-1}}^{N_n} d_i^n \geqslant \tfrac{1}{4} \Big(\sum_{j=1}^{s} c_j + \sum_{\substack{k=1 \\ k \text{ odd}}}^{q} e_k - \sum_{\substack{k=1 \\ k \text{ even}}}^{q} e_k \Big). \tag{2.15}$$

Proof. Letting $c_j = d_{h(j)}^n$
we have by lemma 2.3

$$\sum_{i=0}^{h(1)} {}^{*} d_i^n \geqslant \tfrac{1}{4} c_1, \qquad \sum_{i=h(s)}^{N_n+1} {}^{*} d_i^n \geqslant \tfrac{1}{4} c_s; \tag{2.16}$$

and by lemma 2.2

$$\sum_{i=h(1)}^{h(\alpha(1))} {}^{*} d_i^n \geqslant \tfrac{1}{4} \sum_{j=2}^{\alpha(1)} c_j = \tfrac{1}{4} \sum_{j=2}^{\alpha(1)-1} c_j + \tfrac{1}{4} e_1, \qquad \sum_{i=h(\alpha(q))}^{h(s)} {}^{*} d_i^n \geqslant \tfrac{1}{4} \sum_{j=\alpha(q)}^{s-1} c_j. \tag{2.17}$$

Again by lemma 2.2 if k is odd, $1 \leqslant k < q$,

$$\sum_{i=h(\alpha(k))}^{h(\alpha(k+1))} {}^{*} d_i^n \geqslant \tfrac{1}{4} \sum_{j=\alpha(k)}^{\alpha(k+1)-1} c_j \tag{2.18}$$

while if k is even, $1<k<q$,

$$
(2.19)\quad \left\{\begin{aligned} \sideset{}{^*}\sum_{i=h(\alpha(k))}^{h(\alpha(k+1))} d_i^n &\geqslant \tfrac{1}{4} \sum_{j=\alpha(k)+1}^{\alpha(k+1)} c_j = \tfrac{1}{4}\Big(\sum_{j=\alpha(k)}^{\alpha(k+1)-1} c_j - c_{\alpha(k)} + c_{\alpha(k+1)}\Big) \\ &= \tfrac{1}{4} \sum_{j=\alpha(k)}^{\alpha(k+1)-1} c_j - \tfrac{1}{4} e_k + \tfrac{1}{4} e_{k+1}. \end{aligned}\right.
$$

Now, adding all the inequalities (2.18) and (2.19), we obtain

$$
(2.20)\quad \left\{\begin{aligned} \sideset{}{^*}\sum_{i=h(\alpha(1))}^{h(\alpha(q))} d_i^n &= \sum_{k=1}^{q-1} \sideset{}{^*}\sum_{i=h(\alpha(k))}^{h(\alpha(k+1))} d_i^n \\ &\geqslant \tfrac{1}{4} \sum_{k=1}^{q-1} \sum_{j=\alpha(k)}^{\alpha(k+1)-1} c_j - \tfrac{1}{4} \sum_{\substack{k=1\\ k \text{ even}}}^{q} e_k + \tfrac{1}{4} \sum_{\substack{k=3\\ k \text{ odd}}}^{q} e_k \\ &= \tfrac{1}{4} \sum_{j=\alpha(1)}^{\alpha(q)-1} c_j - \tfrac{1}{4} \sum_{\substack{k=1\\ k \text{ even}}}^{q} e_k + \tfrac{1}{4} \sum_{\substack{k=3\\ k \text{ odd}}}^{q} e_k. \end{aligned}\right.
$$

Now recalling (2.11) and invoking the inequalities (2.16), (2.17) and (2.20) we obtain

$$
\begin{aligned}
\sum_{\substack{i=1\\ L(d_i^n)=(n+1)^*-1}}^{N_n} d_i^n &= \sideset{}{^*}\sum_{i=0}^{N_n+1} d_i^n = \sideset{}{^*}\sum_{i=0}^{h(1)} d_i^n + \sideset{}{^*}\sum_{i=h(1)}^{h(\alpha(1))} d_i^n + \sideset{}{^*}\sum_{i=h(\alpha(1))}^{h(\alpha(q))} d_i^n \\
&\quad + \sideset{}{^*}\sum_{i=h(\alpha(q))}^{h(s)} d_i^n + \sideset{}{^*}\sum_{j=h(s)}^{N_n+1} d_i^n \\
&\geqslant \tfrac{1}{4} c_1 + \tfrac{1}{4} \sum_{j=2}^{\alpha(1)-1} c_j + \tfrac{1}{4} e_1 + \tfrac{1}{4} \sum_{j=\alpha(1)}^{\alpha(q)-1} c_j - \tfrac{1}{4} \sum_{\substack{k=1\\ k \text{ even}}}^{q} e_k + \tfrac{1}{4} \sum_{\substack{k=3\\ k \text{ odd}}}^{q} e_k \\
&\qquad + \tfrac{1}{4} \sum_{j=\alpha(q)}^{s-1} c_j + \tfrac{1}{4} c_s \\
&= \tfrac{1}{4} \sum_{j=1}^{s} c_j - \tfrac{1}{4} \sum_{\substack{k=1\\ k \text{ even}}}^{q} e_k + \tfrac{1}{4} \sum_{\substack{k=1\\ k \text{ odd}}}^{q} e_k.
\end{aligned}
$$

This completes the proof of lemma 2.5.

THEOREM 2.2. *If $c_1, c_2, \ldots, c_s$ is a subsequence of $d_1^n, d_2^n, \ldots, d_{N_n}^n$ then*

$$
(2.21)\qquad \sum_{j=1}^{s-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=1}^{s} c_j - 8\,(2\varepsilon_n + 2^{-n^*}).
$$

Proof. Let $e_1, e_2, \ldots, e_q$ be a subsequence of $c_1, c_2, \ldots, c_s$ formed as in the construction preceding lemma 2.5. Then since $e_1, e_2, \ldots, e_q$ is a subsequence of $c_1, c_2, \ldots, c_s$ we have

$$
(2.22)\qquad \sum_{=1}^{s-1} |c_j - c_{j+1}| \geqslant \sum_{k=1}^{q-1} |e_k - e_{k+1}|.
$$

Moreover, since for even k

$$L(e_k) < L(e_{k-1}) \quad \text{and} \quad L(e_k) < L(e_{k+1}),$$

we see that for even k

$$e_k > e_{k-1} \quad \text{and} \quad e_k > e_{k+1}$$

so that

$$\text{(2.23)} \qquad \begin{cases} \sum_{k=1}^{q-1} |e_k - e_{k+1}| = 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{q} e_k - 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{q} e_k + e_1 + e_q \\ \geqslant 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{q} e_k - 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{q} e_k. \end{cases}$$

Now we rewrite the inequality (2.15) in the form

$$\text{(2.24)} \qquad 2 \sum_{\substack{k=1 \\ k \text{ even}}}^{q} e_k - 2 \sum_{\substack{k=1 \\ k \text{ odd}}}^{q} e_k \geqslant 2 \sum_{j=1}^{s} c_j - 8 \sum_{\substack{i=1 \\ L(d_i^n) = (n+1)^* - 1}}^{N_n} d_i^n.$$

Inequalities (2.22), (2.23) and (2.24) together with remark 2.3 yield

$$\text{(2.25)} \qquad \sum_{j=1}^{s-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=1}^{s} c_j - 8 \sum_{\substack{i=1 \\ L(d_i^n) = (n+1)^* - 1}}^{N_n} d_i^n \geqslant 2 \sum_{j=1}^{s} c_j - 8(2\varepsilon_n + 2^{-n^*})$$

which was to be proved.

THEOREM 2.3. *If $c_1, c_2, \ldots, c_m$ is a finite subsequence of $d_1, d_2, \ldots,$ then*

$$\sum_{j=1}^{m-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=1}^{m} c_j - 16\varepsilon - 48a^* - 16 \max_{x>0} (N(x)/x)$$

where a^* denotes the $\max\limits_{k=1}^{\infty} a_k$.

Proof. We write

$$\text{(2.26)} \qquad \sum_{j=1}^{m-1} |c_j - c_{j+1}| = \sum_{j=1}^{m-1}{}' |c_j - c_{j+1}| + \sum_{n=0}^{\infty} \sum_{j=v_n}^{w_n - 1} |c_j - c_{j+1}|$$

where $\sum'$ is extended over those terms for which c_j and c_{j+1} are not drawn from the same block $d_1^n, d_2^n, \ldots, d_{N_n}^n$ and v_n and w_n are the integers such that c_j is drawn from the block $d_1^n, d_2^n, \ldots, d_{N_n}^n$ if and only if $v_n \leqslant j \leqslant w_n$, the sum $\sum\limits_{j=v_n}^{w_n-1} |c_j - c_{j+1}|$ being considered empty when v_n and w_n fail to exist. Certainly,

$$\text{(2.27)} \qquad \sum_{j=1}^{m-1}{}' |c_j - c_{j+1}| \geqslant 0$$

while

$$\text{(2.28)} \qquad \sum_{j=1}^{m-1}{}' (c_j + c_{j+1}) \leqslant 2 \sum_{n=0} 2^{-n^*} \leqslant 4 \cdot 2^{-0^*}$$

since the sum on the left contains at most 2 terms from any block $d_1^n, d_2^n, \ldots, d_{N_n}{}^n$, and each term in such a block is less than or equal to 2^{-n^*}. By definition 2.1 and (2.8) we see that

$$a^* > 2^{-(0^*+1)} = {}^1/_2\, 2^{-0^*}. \tag{2.29}$$

Combining (2.27), (2.28) and (2.29) we obtain

$$\left\{\begin{aligned} \sum_{j=1}^{m-1}{}' |c_j - c_{j+1}| \geqslant 0 &\geqslant 2 \sum_{j=1}^{m-1}{}' (c_j + c_{j+1}) - 8\cdot 2^{-0^*} \\ &\geqslant 2 \sum_{j=1}^{m-1}{}' (c_j + c_{j+1}) - 16a^*. \end{aligned}\right. \tag{2.30}$$

By Theorem 2.2 for $n = 0, 1, 2, \ldots$

$$\sum_{j=v_n}^{w_n-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=v_n}^{w_n} c_j - 8(2\varepsilon_n + 2^{-n^*})$$

so that by the last formula and (2.7) and (2.8),

$$\left\{\begin{aligned} \sum_{n=0}^{\infty} \sum_{j=v_n}^{w_n-1} |c_j - c_{j+1}| &\geqslant \sum_{n=0}^{\infty} 2 \sum_{j=v_n}^{w_n} c_j - 16 \sum_{n=0}^{\infty} \varepsilon_n - 8 \sum_{n=0}^{\infty} 2^{-n^*} \\ &\geqslant 2 \sum_{n=0}^{\infty} \sum_{j=v_n}^{w_n} c_j - 16\varepsilon_0 - 16\varepsilon - 16\cdot 2^{-0^*} \\ &\geqslant 2 \sum_{n=0}^{\infty} \sum_{j=v_n}^{w_n} c_j - 16 \max_{x>0}\, (N(x)/x) - 16\varepsilon - 32a^*. \end{aligned}\right. \tag{2.31}$$

Now (2.26), (2.30) and (2.31) yield

$$\begin{aligned} \sum_{=1}^{m-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=1}^{m-1}{}' (c_j + c_{j+1}) &+ 2 \sum_{n=0}^{\infty} \sum_{j=v_n}^{w_n} c_j - 16a^* \\ &- 32a^* - 16\varepsilon - 16 \max_{x>0}\, (N(x)/x) \\ \geqslant 2 \sum_{j=1}^{m} c_j - 48a^* - 16\varepsilon &- 16 \max_{x>0}\, (N(x)/x). \end{aligned}$$

THEOREM 2.4. *If $a_1, a_2, a_3, \ldots$ is a sequence with $a_n > 0$ for $n = 1, 2, \ldots$, $a_n \to 0$ as $n \to \infty$ and $N(x) = o(x)$, then for any $\eta > 0$ there is a complete rearrangement $b_1, b_1, b_3, \ldots$ of $a_1, a_2, a_3, \ldots$ such that for any subsequence $c_1, c_2, c_3, \ldots$ of $b_1, b_2, b_3, \ldots$,*

$$\sum_{j=1}^{m-1} |c_j - c_{j+1}| \geqslant 2 \sum_{j=1}^{m} c_j - 48a^* - 16 \max_{x>0}\, (N(x)/x) - \eta, \tag{2.32}$$

where $a^* = \max_{k=1}^{\infty} a_k$.

Proof. Construct the sequence $d_1, d_2, d_3, \ldots$ as before taking $\varepsilon = \eta/16$. Now let $b_1, b_2, b_3, \ldots$ be that subsequence of $d_1, d_2, d_3, \ldots$ consisting of just those terms of $d_1, d_2, d_3, \ldots$ which were chosen from $a_1, a_2, a_3, \ldots$.

Since (see Remark 2.1) $d_1, d_2, d_3, \ldots$ exhausts the terms of $a_1, a_2, a_3, \ldots$ we note that $b_1, b_2, b_3, \ldots$ is a complete rearrangement of $a_1, a_2, a_3, \ldots$. Now $c_1, c_2, c_3, \ldots$ being a subsequence of $b_1, b_2, b_3, \ldots$ is also a subsequence of $d_1, d_2, d_3, \ldots$ and the conclusion of Theorem 2.3 holds.

COROLLARY. *If* $a_1, a_2, a_3, \ldots$ *is a sequence with* $a_n > 0$, $n = 1, 2, 3, \ldots$, $a_n \to 0$ *as* $n \to \infty$, $\sum_{n=1}^{\infty} a_n = \infty$, *and* $N(x) = o(x)$ *then there is a complete rearrangement* $b_1, b_2, b_3, \ldots$ *of* $a_1, a_2, a_3, \ldots$ *such that whenever* $c_1, c_2, c_3, \ldots$ *is a subsequence of* $b_1, b_2, b_3, \ldots$ *with* $\sum_{n=1}^{\infty} c_n = \infty$ *then also* $\sum_{n=1}^{\infty} |c_n - c_{n+1}| = \infty$.

3. *Other Differences*

DEFINITION 3.1. We say that a sequence $a_1, a_2, \ldots$ is admissible if all its terms are positive, $a_n \to 0$ as $n \to \infty$ and $\sum_{n=1}^{\infty} a_n = \infty$.

In lemmas 3.1, 3.2 and 3.3 we let $a_1, a_2, \ldots$ denote an admissible sequence; χ an integer $\geqslant 1$; M an integer $\geqslant 0$; $\tau_0, \tau_1, \ldots$ numbers > 1. And as shown to be possible in Lemma 1 of [I] we let $b_1, \ldots, b_\mu$ denote a finite rearrangement of $a_1, a_2, \ldots$ and $l(0), l(1), \ldots, l(\mu+1)$ denote integers with the following properties:

I $\quad l(0) = l(\mu+1) = -1$; $l(k) \geqslant 0$ for $1 \leqslant k \leqslant \mu$; $\max_{1 \leqslant k \leqslant \mu} l(k) = M$.

II $\quad$ There is exactly one integer k for which $l(k) = 0$, and for this k, $b_k = a_\chi$.

III $\quad$ If i and j are positive integers $\leqslant \mu$ with $l(i) < l(j)$ then $b_i > b_j$.

IV $\quad$ If $0 \leqslant i < j \leqslant \mu+1$, if $0 \leqslant r = \max\,(l(i), l(j)) < M$, and if $l(k) > r$ for $k = i+1, i+2, \ldots, j-1$, then

$$
{}^1\!/_2 \sum_{\substack{k=i,j \\ l(k)=r}} b_k \leqslant \sum_{\substack{i<k<j \\ l(k)=r+1}} b_k < \tau_r/2 \sum_{\substack{k=i,j \\ l(k)=r}} b_k.
$$

In addition q denotes a positive integer and $\alpha_0, \alpha_1, \ldots, \alpha_q$ denote real or complex numbers not all equal to zero. Finally $c_1, c_2, \ldots, c_p$ denotes an arbitrary finite subsequence of $b_1, b_2, \ldots, b_\mu$ and $h(k)$ is the increasing function on the integers $1, 2, \ldots, p$ such that $c_k = b_{h(k)}$, $k = 1, 2, \ldots, p$.

DEFINITION 3.2. Let us consider a sum of the form

$$
\sum_{\substack{k=m \\ P(k)}}^{n} b_k
$$

where m and n are integers with $m < n$ and where $P(k)$ means that only the terms b_k are to be included for which k satisfies a certain condition $P(k)$. Then we define

$$
\sum_{\substack{k=m \\ P(k)}}^{n}{}^{*}\, b_k = \sum_{\substack{k=m+1 \\ P(k)}}^{n-1} b_k + {}^1\!/_2 \sum_{\substack{k=m,n \\ P(k)}} b_k.
$$

LEMMA 3.1. *For each integer n with $1 \leqslant n \leqslant p-q$ we have*

$$|\alpha_0 c_n + \ldots + \alpha_q c_{n+q}| \geqslant \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 4(q+1) \max_{k=0}^{q} |\alpha_k| \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)}{}^{*} b_k.$$

Proof. Choose integers n^* and n_* with $0 \leqslant n^* \leqslant q$, $0 \leqslant n_* \leqslant q$, and such that

$$c_{n+n^*} = \max_{j=0}^{q} c_{n+j}; \quad c_{n+n_*} = \max_{\substack{j=0\\ j\neq n^*}}^{q} c_{n+j}.$$

We now divide the remainder of the proof into two parts.

PART I

$$|\alpha_0 c_n + \ldots + \alpha_q c_{n+q}| \geqslant \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 2(q+1) \max_{k=0}^{q} |\alpha_k| c_{n+n_*}.$$

Proof. $|\alpha_0 c_n + \ldots + \alpha_q c_{n+q}|$

$$\geqslant |\alpha_{n^*}|\, c_{n+n^*} - \sum_{\substack{k=0\\ k\neq n^*}}^{q} |\alpha_k| c_{n+k}$$

$$\geqslant \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 2 \sum_{\substack{k=0\\ k\neq n^*}}^{q} |\alpha_k| c_{n+k}$$

$$\geqslant \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 2 \max_{k=0}^{q} |\alpha_k| \sum_{\substack{k=0\\ k\neq n^*}}^{q} c_{n+k}$$

$$\geqslant \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 2 \max_{k=0}^{q} |\alpha_k| (q+1)\, c_{n+n_*}.$$

PART II

$$c_{n+n_*} \leqslant 2 \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)}{}^{*} b_k.$$

Proof. Using the relation

$$b_{h(n+n_*)} = c_{n+n_*} \leqslant c_{n+n^*} = b_{h(n+n^*)}$$

we see that $l(h(n+n_*)) \geqslant l(h(n+n^*))$. Now assume that $n_* < n^*$ so that $h(n+n_*) < h(n+n^*)$. Applying Lemma 9 of [I] we see that

$$\tfrac{1}{2}\, c_{n+n_*} \leqslant \sum_{\substack{k=h(n+n_*)\\ l(k)=M}}^{h(n+n^*)}{}^{*} b_k \leqslant \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)}{}^{*} b_k.$$

PART III. The Lemma now follows by substituting the result of Part II into that of Part I.

LEMMA 3.2.

$$\sum_{n=1}^{p-q} \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)}{}^{*} b_k \leqslant q \tau_0 \tau_1 \ldots \tau_{M-1}\, a_{\chi}.$$

Proof. First observe that

$$\sum_{n=1}^{p-q} \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)} {}^{*} b_k = \sum_{s=1}^{q} \sum_{t=0}^{[(p-s)/q]-1} \sum_{\substack{k=h(tq+s)\\ l(k)=M}}^{h((t+1)q+s)} {}^{*} b_k.$$

From the definition of $\sum^*$ we see that for each $s=1, 2, \ldots, q$,

$$\sum_{t=0}^{[(p-s)/q]-1} \sum_{\substack{k=h(tq+s)\\ l(k)=M}}^{h((t+1)q+s)} {}^{*} b_k = \sum_{\substack{k=h(s)\\ l(k)=M}}^{h([(p-s)/q]q+s)} {}^{*} b_k \leqslant \sum_{\substack{k=h(s)\\ l(k)=M}}^{h(p)} {}^{*} b_k \leqslant \sum_{\substack{k=0\\ l(k)=M}}^{\mu+1} {}^{*} b_k$$
$$= \sum_{\substack{k=1\\ l(k)=M}}^{\mu} b_k < \tau_0\tau_1\ldots\tau_{M-1}\, a_\chi$$

by Lemma 4 of [I]. Consequently

$$\sum_{s=1}^{q} \sum_{t=0}^{[(p-s)/q]-1} \sum_{k=h(tq+s}^{h((t+1)p+s)} {}^{*} b_k \leqslant q\tau_0\tau_1\ldots\tau_{M-1}\, a_\chi$$

which gives the desired result.

Lemma 3.3.

$$\sum_{n=1}^{p-q} |\alpha_0 c_n + \ldots + \alpha_q c_{n+q}| \geqslant \sum_{k=0}^{q} |\alpha_k| \sum_{n=1}^{p} c_n - q(q+1) \max_{k=0}^{p} |\alpha_k| a_\chi (1+4\tau_0\tau_1\ldots\tau_{M-1}).$$

Proof

$$\sum_{n=1}^{p-q} \sum_{k=0}^{q} |\alpha_k| c_{n+k} = \sum_{m=1}^{p} \sum_{\substack{0\leqslant k\leqslant q\\ m+q-p\leqslant k<m}} |\alpha_k| c_m = \sum_{m=1}^{p} c_m \sum_{\substack{0\leqslant k\leqslant q\\ m+q-p\leqslant k<m}} |\alpha_k|$$

$$\geqslant \sum_{m=1}^{p} (|\alpha_0| + \ldots + |\alpha_q|) c_m - \max_{m=1}^{p} c_m \max_{k=1}^{q} |\alpha_k| \Big(\sum_{m=1}^{q} \sum_{k=0}^{m-1} 1 + \sum_{m=p-q}^{p} \sum_{k=m+q-p+1}^{q} 1\Big)$$

$$\geqslant \sum_{m=1}^{p} (|\alpha_0| + \ldots + |\alpha_q|) c_m - a_\chi \max_{k=1}^{q} |\alpha_k|\, q(q+1).$$

Now using Lemmas 3.1 and 3.2 together with the above we find

$$\sum_{n=1}^{p-q} |\alpha_0 c_n + \ldots + \alpha_q c_{n+q}| \geqslant \sum_{n=1}^{p-q} \sum_{k=0}^{q} |\alpha_k| c_{n+k} - 4(q+1) \max_{k=1}^{q} |\alpha_k| \sum_{n=1}^{p-q} \sum_{\substack{k=h(n)\\ l(k)=M}}^{h(n+q)} {}^{*} b_k$$

$$\geqslant \sum_{n=1}^{p} (|\alpha_0| + \ldots + |\alpha_q|) c_n - a_\chi \max_{k=1}^{p} |\alpha_k| q(q+1) - 4\, q(q+1) \max_{k=1}^{p} |\alpha_k| a_\chi \tau_0\tau_1\ldots\tau_{M-1}$$

$$= \sum_{n=1}^{p} (|\alpha_0| + \ldots + |\alpha_q|) c_n - q(q+1) \max_{k=1}^{q} |\alpha_k| (1 + 4\, \tau_0\tau_1\ldots\tau_{M-1})$$

which was desired.

Theorem 3.1. *If $a_1, a_2, \ldots$ is admissible, and $K > 5$ then there exists an admissible rearrangement $b_1, b_2, \ldots$ of $a_1, a_2, \ldots$ having the following* *property: if we choose a positive integer q and numbers $\alpha_1, \alpha_2, \ldots, \alpha_q$ then for any subsequence $c_1, c_2, \ldots$ of $b_1, b_2, \ldots$ we have*

$$\sum_{k=1}^{n-q} |\alpha_0 c_k + \ldots + \alpha_q c_{q+k}| \geqslant (|\alpha_1| + \ldots + |\alpha_q|) \sum_{k=1}^{n} c_k - CKb^*$$

where b^* denotes the largest of the terms $b_1, b_2, \ldots$. Here $C = q(q+1) \max\limits_{k=0}^{q} |\alpha_k|$.

REMARK. The rearrangement $b_1, b_2, \ldots$ which will be obtained here and shown to have the desired property is precisely that employed in theorem 2 of [I]. The proof of this theorem parallels that of theorem 2 of [I].

Proof. Let $\gamma_1, \gamma_2, \ldots$ be a sequence of positive numbers with $\sum\limits_{k=1}^{\infty} \gamma_k = \infty$. Choose a term a_{χ_1} of $a_1, a_2, \ldots$; let λ be a number with $1 < \lambda < K/5$ and let $\beta_1, \beta_2, \ldots$ be a sequence of positive numbers with

$$\beta_1 = a_{\chi_1} \text{ and } \sum_{k=1}^{\infty} \beta_k < \lambda a_{\chi_1}.$$

Let τ be a number with $1 < \tau < \frac{1}{4}(K/\lambda - 1)$ and let $\tau_0, \tau_1, \ldots$ be numbers > 1 such that

$$\prod_{k=0}^{\infty} \tau_k = \tau.$$

Now for each positive integer m find a finite rearrangement $b_1^{(m)}, \ldots, b_{\mu_m}^{(m)}$ satisfying the condition of lemma 1 of [I] with χ replaced by χ_m, μ replaced by μ_m and M replaced by M_m. The integer χ_1 has already been chosen; choose M_m so large that

$$\sum_{k=1}^{\mu_m} b_k^{(m)} \geqslant \gamma_m.$$

This is possible since by lemma 5 of [I]

$$\sum_{k=1}^{\mu_m} b_k^{(m)} \geqslant (M_m + 1)\, a_{\chi_m}.$$

After M_m has been chosen we choose χ_{m+1} so that

$$a_{\chi_{m+1}} < \min_{1 \leqslant k \leqslant \mu_m} b_k \quad \text{and} \quad a_{\chi_{m+1}} \leqslant \beta_{m+1}.$$

Consider the sequence

$$b_1^{(1)}, b_2^{(2)}, \ldots, b_{\mu_1}^{(1)}, b_2^{(2)}, \ldots, b_{\mu_2}^{(2)}, b_1^{(3)}, \ldots$$

and designate the terms in order $b_1, b_2, \ldots$. We shall show that this sequence has the required property.

Now let $c_1, c_2, \ldots, c_n$ be a finite subsequence of $b_1, b_2, \ldots$. Let

$$c_i = b_{h_i} \quad \text{for} \quad i = 1, 2, \ldots, n.$$

Let $0=i_0\leqslant i_1\leqslant i_2\leqslant\ldots\leqslant i_p=n$ be the integers such that for

$$k=i_{m-1}+1,\ i_{m-1}+2,\ \ldots,\ i_m$$

we have

$$\mu_1+\mu_2+\ldots+\mu_{m-1}+1\leqslant h_k\leqslant\mu_1+\mu_2+\ldots+\mu_m.$$

This means that the terms c_k with $i_{m-1}<k<i_m$ are the terms of $c_1, c_2, \ldots, c_n$ to be found among the terms $b_1{}^{(m)}, b_2{}^{(m)}, \ldots, b_{\mu_m}{}^{(m)}$. Under these conditions we see that

$$\sum_{k=1}^{n-q}|\alpha_0 c_k+\ldots+\alpha_q c_{q+k}|\geqslant\sum_{m=1}^{p}\sum_{k=i_{m-1}+1}^{i_m-q}|\alpha_1 c_k+\ldots+\alpha_q c_{q+k}|. \tag{3.1}$$

Now lemma 3.3 assures us that

$$\left\{\begin{aligned}&\sum_{k=i_{m-1}+1}^{i_m-q}|\alpha_0 c_k+\ldots+\alpha_q c_{q+k}|\geqslant(|\alpha_1|+\ldots+|\alpha_q|)\sum_{k=i_{m-1}+1}^{i_m}c_k\\&\quad-q(q+1)\max_{k=0}^{p}|\alpha_k|\ a_{\chi_m}(4\tau_0\ldots\tau_{M_m-1}+1)\\&\quad\geqslant(|\alpha_0|+\ldots+|\alpha_q|)\sum_{k=i_{m-1}+1}^{i_m}c_k-C(4\tau+1)a_{\chi_m}.\end{aligned}\right. \tag{3.2}$$

Relation (3.1) combined with (3.2) yields

$$\begin{aligned}\sum_{k=1}^{n-q}|\alpha_0 c_k+\ldots+\alpha_q c_{q+k}|&\geqslant\sum_{m=1}^{p}\{(|\alpha_1|+\ldots+|\alpha_q|)\sum_{k=i_{m-1}+1}^{i_m}c_k-C(4\tau+1)a_{\chi_m}\}\\&=(|\alpha_0|+\ldots+|\alpha_q|)\sum_{k=1}^{n}c_k-C(4\tau+1)\sum_{m=1}^{p}a_{\chi_m}\\&\geqslant(|\alpha_0|+\ldots+|\alpha_q|)\sum_{k=1}^{n}c_k-C(4\tau+1)\lambda b^*\\&>(|\alpha_0|+\ldots+|\alpha_q|)\sum_{k=1}^{n}c_k-CKb^*.\end{aligned}$$

The fact that $b_1, b_2, \ldots$ is admissible follows from

$$\sum_{k=1}^{\infty}b_k\geqslant\sum_{m=1}^{\infty}\gamma_m=\infty.$$

COROLLARY 1. For every subsequence

$$c_1, c_2, \ldots \quad\text{of}\quad b_1, b_2, \ldots \text{ with } \sum_{n=1}^{\infty}c_n=\infty$$

we have

$$\sum_{k=1}^{\infty}|\alpha_0 c_k+\ldots+\alpha_q c_{q+k}|=\infty.$$

Proof. By theorem 3.1 we have

$$\begin{aligned}\sum_{k=1}^{\infty}|\alpha_0 c_k+\ldots+\alpha_q c_{q+k}|&=\lim_{n\to\infty}\sum_{k=1}^{n-p}|\alpha_1 c_k+\ldots+\alpha_q c_{q+k}|\\&\geqslant\lim_{n\to\infty}\{(|\alpha_1|+\ldots+|\alpha_q|)\sum_{k=1}^{n}c_k-CKb^*\}=\infty.\end{aligned}$$

COROLLARY 2. If $c_1, c_2, \ldots, c_n$ is a finite subsequence of $b_1, b_2, \ldots$ then the sum, S, of the absolute values of the p^{th} differences of the numbers $c_1, c_2, \ldots, c_n$ satisfies

$$S > 2^p \sum_{k=1}^{n} c_k - p(p+1)\binom{p}{[p/2]} Kb^*.$$

Proof. By definition

$$S = \sum_{k=1}^{n-p} \left| \binom{p}{0} c_k - \binom{p}{1} c_{1+k} + \ldots \pm \binom{p}{p} c_{p+k} \right|$$

and this sum is by theorem 3.1 greater than

$$\left\{\binom{p}{0} + \binom{p}{1} + \ldots + \binom{p}{p}\right\} \sum_{k=1}^{n} c_k - p(p+1) \max_{k=0}^{p} \binom{p}{k} Kb^*.$$

The fact that

$$\left\{\binom{p}{0} + \binom{p}{1} + \ldots + \binom{p}{p}\right\} = 2^p \text{ and } \max_{k=0}^{p} \binom{p}{k} = \binom{p}{[p/2]}$$

completes the proof.

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